

# Birkhoff strata of the Grassmannian $\text{Gr}^{(2)}$ : Algebraic curves

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## Abstract

Algebraic varieties and curves arising in Birkhoff strata of the Sato Grassmannian  $\text{Gr}^{(2)}$  are studied. It is shown that the big cell  $\Sigma_0$  contains the tower of families of the normal rational curves of all odd orders. Strata  $\Sigma_{2n}$ ,  $n = 1, 2, 3, \dots$  have hyperelliptic subsets  $W_{2n}$  with the points containing hyperelliptic curves of genus  $n$  and their coordinate rings. Strata  $\Sigma_{2n+1}$ ,  $n = 0, 1, 2, 3, \dots$  contain  $(2m+1, 2m+3)$ -plane curves for  $n = 2m, 2m-1$  ( $m \geq 2$ ) and  $(3, 4)$  and  $(3, 5)$  curves in  $\Sigma_3, \Sigma_5$  respectively. Curves in the strata  $\Sigma_{2n+1}$  have zero genus.

## 1 Introduction

Grassmannian  $\text{Gr}^{(2)}$  is a very important specialization of the universal Sato Grassmannian [1]. The most known its appearance is due to the connection with the theory of the KdV equation [2, 3]. The present paper is devoted to the study of the Grassmannian  $\text{Gr}^{(2)}$  within the framework proposed recently in [4]. The main idea of this approach is to analyze algebro-geometric structures arising in Sato Grassmannian, in our case in the Birkhoff strata of  $\text{Gr}^{(2)}$ , without any *a priori* reference to any integrable system.

Recall that Sato Grassmannian  $\text{Gr}$  can be viewed as the set of closed vector subspaces in the infinite dimensional set of all formal Laurent series with coefficients in  $\mathbb{C}$  with certain special properties (see e.g. [2, 3]). Each subset  $W \subset \text{Gr}$  contains points possessing an algebraic basis  $(w_0(z), w_1(z), w_2(z), \dots)$  where

$$w_n = \sum_{k=-\infty}^n a_k z^k \quad (1)$$

of finite order  $n$ . Grassmannian  $\text{Gr}$  is a connected Banach manifold which exhibits a stratified structure [2, 3], i.e.  $\text{Gr} = \bigcup_S \Sigma_S$  where the stratum  $\Sigma_S$  is a subset in  $\text{Gr}$  formed by elements of the form (1) such that possible values  $n$  are given by the infinite set  $S = \{s_0, s_1, s_2, \dots\}$  of integers  $s_n$  with  $s_0 < s_1 < s_2 < \dots$  and  $s_n = n$  for large  $n$ . Big cell  $\Sigma_0$  corresponds to  $S = \{0, 1, 2, \dots\}$ . Other strata are associated with the sets  $S$  different from  $S_0$ .

$\text{Gr}^{(2)}$  is the subset of elements  $W$  of  $\text{Gr}$  obeying the condition  $z^2 \cdot W \subset W$  [2, 3]. This condition imposes strong constraints on the Laurent series and on the structure of the strata. Namely, Birkhoff stratum  $\Sigma_S$  in  $\text{Gr}^{(2)}$  corresponds to the sets  $S$  such that  $S + 2 \subset S$ , i.e. all possible  $S$  having the form [2, 3]

$$S_m = \{-m, -m+2, -m+4, \dots, m, m+1, m+2, \dots\} \quad (2)$$

with  $m = 0, 1, 2, \dots$ . Codimension of  $\Sigma_m$  is  $m(m+1)/2$ . One has  $\text{Gr}^{(2)} = \bigcup_{m \geq 0} \Sigma_m$ .

In this paper, using the properties of the Birkhoff strata  $\text{Gr}^{(2)}$ , we show that the big cell  $\Sigma_0$  contains a maximal closed subset  $W_0$  which geometrically is a tower of infinite families of rational normal (Veronese) curves of all odd orders. It is demonstrated that the strata  $\Sigma_{2n}$ ,  $n = 1, 2, \dots$  contain subsets  $W_{2n}$  closed with respect to pointwise multiplication if the coefficients of Laurent series  $w_n$  obey certain associativity constraints. Geometrically the subsets  $W_{2n}$  represent infinite families of coordinate rings for the hyperelliptic curves of genus  $n$ . Each point of the subset  $W_{2n}$  contains hyperelliptic curves and its

coordinate rings. Then it is shown that the strata  $\Sigma_3$  and  $\Sigma_5$  contain  $(3, 4)$  and  $(3, 5)$  degenerate plane curves respectively. In the strata  $\Sigma_{2m+1}$ ,  $m \geq 2$  one has families of  $(2m+1, 2m+3)$  plane curves of zero genus.

In the second part of this work [5] the tangent cohomology of the subsets  $W_n$  and associated integrable systems of hydrodynamical type will be studied.

The paper is organized as follows. The big cell is discussed in section 2. Stratum  $\Sigma_1$  is considered in section 3. Closed subsets  $W_2$  in the stratum  $\Sigma_2$  and corresponding elliptic curves are studied in section 4. Stratum  $\Sigma_3$  and associated  $(3, 4)$  curves are analysed in section 5. Section 6 is devoted to general strata  $\Sigma_{2n}$ ,  $(n = 2, 3, 4, \dots)$ . Stratum  $\Sigma_5$  and the generic strata  $\Sigma_{2n+1}$ ,  $(n = 3, 4, \dots)$  are discussed in section 7.

## 2 Big cell

The principal stratum  $\Sigma_0$  for which  $S = \{0, 1, 2, \dots\}$  (called also big cell) is a dense open set and it has codimension zero [2, 3]. It possesses a canonical basis  $(p_0, p_1, p_2, \dots)$  where

$$p_i(z) = z^i + \sum_{k \geq 1} \frac{H_k^i}{z^k}, \quad i = 0, 1, 2, \dots \quad (3)$$

with arbitrary  $H_k^i$ .

Accordingly to the approach proposed in [4] we first look for a subset  $W_0 \subset \Sigma_0$  closed with respect to multiplication. Similar to the big cell in the general Gr one has

**Lemma 2.1** *Laurent series (3) at fixed  $H_k^j$  obey the condition  $z^2 W_0 \subset W_0$  and the equations*

$$p_j(z)p_k(z) = \sum_{l \geq 0} C_{jk}^l p_l(z) \quad (4)$$

if and only if

$$\begin{aligned} H_i^{2n} &= 0, & i = 1, 2, 3, \dots, n = 0, 1, 2, \dots, \\ H_{2i}^{2n+i} &= 0, & i = 1, 2, 3, \dots, n = 0, 1, 2, \dots \end{aligned} \quad (5)$$

and

$$\begin{aligned} H_{2(k+n)+1}^{2m+1} - H_{2k+1}^{2(m+n)+1} - \sum_{s=0}^{n-1} H_{2s+1}^{2m+1} H_{2k+1}^{2(n-s)-1} &= 0, \\ H_{2(k+n)+1}^{2m+1} + H_{2(k+m)+1}^{2n+1} + \sum_{l=0}^{k-1} H_{2l+1}^{2m+1} H_{2(k-l)-1}^{2n+1} &= 0. \end{aligned} \quad (6)$$

The constants  $C_{jk}^l$  are given by

$$\begin{aligned} C_{2n, 2m}^{2l} &= \delta_{m+n}^l, \\ C_{2n, 2m+1}^{2l+1} &= \delta_{m+n}^l + H_{2(n-l)-1}^{2m+1}, \\ C_{2n+1, 2m+1}^{2l} &= \delta_{m+n+1}^l + H_{2(m-l)+1}^{2n+1} + H_{2(n-l)+1}^{2m+1} \end{aligned} \quad (7)$$

and  $p_{2n} = p_2^2 = z^{2n}$ ,  $n \geq 0$ .

An immediate consequence of this lemma is given by the following

**Proposition 2.2** *The subset  $W_0 \subset \Sigma_0$  the elements of which are given by vector spaces with basis  $\langle p_i(z) \rangle_i$  and parameters  $H_k^i$  obeying the constraints (6), is closed with respect to pointwise multiplication  $W_0 \cdot W_0 \subset W_0$ . It is a maximal closed subset in the big cell. This subset  $W_0$  is an infinite family of infinite-dimensional associative commutative algebra with unity  $p_0 = 1$ .*

The last statement follows from the equivalence of equations (6) to the associativity conditions

$$\sum_s C_{ij}^s C_{ks}^r - C_{ik}^s C_{js}^r = 0 \quad (8)$$

for the structure constants  $C_{jk}^l$ .

Relations (4) written explicitly, i.e.

$$\begin{aligned} p_{2n}p_{2m} &= p_{2(m+n)}, \\ p_{2n}p_{2m+1} &= p_{2(m+n)+1} + \sum_{s=0}^{n-1} H_{2s+1}^{2m+1} p_{2(n-s)-1}, \\ p_{2n+1}p_{2m+1} &= p_{2(m+n+1)} + \sum_{s=0}^m H_{2s+1}^{2n+1} p_{2(m-s)} + \sum_{s=0}^n H_{2s+1}^{2m+1} p_{2(n-s)}, \end{aligned} \quad (9)$$

imply that

$$\begin{aligned} z^2 &= p_1^2 - 2H_1^1, \\ p_3 &= p_1^3 - 3H_1^1 p_1, \\ p_5 &= p_1^5 - 5H_1^1 p_1^3 + \frac{15}{2} H_1^1{}^2 p_1, \\ &\dots \end{aligned} \quad (10)$$

or equivalently

$$\lambda = p_1^2 - 2H_1^1, \quad p_{2n+1} = \alpha_n(\lambda) p_1 \quad (11)$$

where  $\lambda = z^2$  and  $\alpha_n(\lambda) = \prod_{s=1}^n \left( \lambda - \frac{H_1^{2(n-s)+1}}{2(n-s)+1} \right)$ .

Similar to [4] one can treat  $\lambda, p_1, p_3, \dots$  as the affine coordinates. So one has the following geometrical interpretation of the subset  $W_0$ .

**Proposition 2.3** *Big cell  $\Sigma_0$  contains an infinite-dimensional algebraic variety  $\Gamma_0$  with the ideal*

$$\langle \lambda - p_1^2 + 2H_1^1, l_1^{(2)}, l_1^{(2)}, \dots \rangle \quad (12)$$

where  $l_n^{(2)} = p_{2n+1} - \alpha(\lambda) p_1$  and the variables  $H_k^j$  obey the constraints (6). This variety  $\Gamma_0$  is an infinite tower of infinite families of rational normal (Veronese) curves of all odd orders.

Formulas (10) represent a canonical parameterization of rational normal curves (see e.g. [6]). For instance, the curves defined by the first two equations (10) is the classical twisted cubic in the three-dimensional space with the coordinates  $(\lambda, p_1, p_3)$ .

There is an infinite set of independent variables among all  $H_k^j$  constrained by conditions (6). A natural set of independent  $H_k^j$  is given by  $H_1^1, H_3^1, H_5^1, \dots$ .

It is also easy to see using (11) that the ideal  $I_0^{(2)}$  contains singular “hyperelliptic” curves of genus zero given by the equations

$$p_{2n+1}^2 = (\lambda + 2H_1^1) \alpha_n(\lambda)^2. \quad (13)$$

Infinite family of algebraic varieties described in Proposition 2.3 is in its turn the algebraic variety in the affine space with coordinates  $p_i$ , ( $i = 1, 2, 3, \dots$ ) and  $H_k^j$ , ( $j, k = 1, 2, 3, \dots$ ) defined by the quadrics

$$f_{jk} = p_j p_k - p_{j+k} - \sum_{s=0}^k H_s^j p_s - \sum_{s=0}^j H_s^k p_s = 0 \quad (14)$$

and equations (6).

We emphasize that an infinite tower of normal rational curves for fixed  $H_k^j$  is in correspondence with a point of the subset  $W_0$ .

### 3 Stratum $\Sigma_1$

The stratum  $\Sigma_1$  is the lowest stratum different from the big cell and it corresponds to  $m = 1$  and  $S = \{-1, 1, 2, \dots\}$ . Due to the absence of zero order element  $w_0$  the canonical basis is of the form

$$p_i(z) = z^i + H_0^i + \sum_{k \geq 1} \frac{H_k^i}{z^k}, \quad i = 1, 2, 3, \dots \quad (15)$$

Since  $(p_{-1})^2 \notin \langle p_i \rangle_{i=-1,1,2,\dots}$  one should consider only  $p_j$  with  $j = 1, 2, 3, \dots$

**Lemma 3.1** *A set  $W_1$  of Laurent series (15) obey the condition  $z^2 \cdot W_1 \subset W_1$  and the equations*

$$p_j(z)p_k(z) = \sum_{l \geq 1} C_{jk}^l p_l(z), \quad j, k = 1, 2, 3, \dots \quad (16)$$

if and only if the parameters  $H_k^j$  satisfy the constraints

$$\begin{aligned} H_0^{2j+k} + H_0^{2j} H_0^{2k} &= 0, \\ H_0^{2(k+j)+1} + H_0^{2k+1} H_0^{2j+1} + \sum_{l=0}^{j-1} H_{2l+1}^{2k+1} H_0^{2(j-l)-1} &= 0, \\ H_{2(j+l)+1}^{2k+1} + H_0^{2j} H_{2l+1}^{2k+1} - H_{2l+1}^{2(k+j)+1} - H_0^{2j} H_{2l+1}^{2k+1} - H_{2(l+j)+1}^{2k+1} - \sum_{s=0}^{j-1} H_{2s+1}^{2k+1} H_{2l+1}^{2(j-s)-1} &= 0, \\ H_0^{2(k+j+1)} + H_0^{2k+1} H_0^{2j+1} + \sum_{s=0}^{k-1} H_0^{2(k-s)} H_{2s+1}^{2j+1} + \sum_{s=0}^{j-1} H_0^{2(j-s)} H_{2s+1}^{2k+1} &= 0, \\ H_{2(l+k)+1}^{2j+1} + H_{2(l+j)+1}^{2k+1} + \sum_{s=0}^{l-1} H_{2s+1}^{2j+1} H_{2(l-s)-1}^{2k+1} - \sum_{s=0}^{k-1} H_{2(l+k)+1}^{2j+1} - \sum_{s=0}^{j-1} H_{2(l+j)+1}^{2k+1} &= 0 \end{aligned} \quad (17)$$

and

$$\begin{aligned} C_{2j,2k}^{2l} &= \delta_{j+k}^l + H_0^{2j} \delta_k^l + H_0^{2k} \delta_j^l, \\ C_{2j,2k+1}^{2l+1} &= \delta_{j+k}^l + H_0^{2j} \delta_k^l + H_0^{2k} \delta_j^l + H_{2(j-l)+1}^{2k+1}, \\ C_{2j+1,2k+1}^{2l+1} &= H_0^{2k+1} \delta_j^l + H_0^{2j+1} \delta_k^l, \\ C_{2j+1,2k+1}^{2l} &= \delta_{j+k+1}^l + H_{2(l+k)+1}^{2j+1} + H_{2(l+j)+1}^{2k+1}. \end{aligned} \quad (18)$$

The analysis of the constraints (17) gives

$$H_i^{2n} = 0, \quad n, i = 1, 2, 3, \dots \quad (19)$$

and

$$H_0^{2n} = -(-H_0^2)^n, \quad n = 1, 2, 3, \dots, \quad (20)$$

i.e.

$$p_{2n}(z) = z^{2n} - (-H_0^2)^n, \quad n = 1, 2, 3, \dots \quad (21)$$

For the elements  $p_{2n+1}$  one instead has

$$\begin{aligned} p_2 &= p_1^2 - 2H_0^1 p_1, \\ p_3 &= p_1^3 - 3H_0^1 p_1^2 - (3H_1^1 - 3H_0^1{}^2) p_1, \\ &\dots \end{aligned} \quad (22)$$

Similar to the big cell one has a subset  $W_1$  in  $\Sigma_1$  closed with respect to multiplication which algebraically is an infinite-dimensional commutative associative algebra  $A_1$  with the structure constants given by (18)

in the basis (15). Geometrically  $W_1$  is an infinite tower of families of rational normal curves of all odd orders passing through the origin  $p_1 = p_2 = p_3 = \dots = 0$ .

The fact that for the stratum  $\Sigma_1$  one has results which are similar to those for big cell is not that surprising. Indeed, taking into account the relations (17), namely

$$2H_1^1 - H_0^2 + H_0^{1^2} = 0, \quad H_0^3 + H_0^1 H_0^2 + H_0^1 H_1^1 = 0 \quad (23)$$

and the formula (21), i.e.  $p_2 = z^2 + H_0^2$ , one can rewrite equations (22) as

$$\begin{aligned} z^2 &= (p_1 - H_0^1)^2 - 2H_1^1, \\ p_3 - H_0^3 &= (p_1 - H_0^1)^3 - 3H_1^1(p_1 - H_0^1). \end{aligned} \quad (24)$$

In the variables

$$\tilde{p}_1 = p_1 - H_0^1, \quad \tilde{p}_3 = p_3 - H_0^3 \quad (25)$$

the equations (24) the first two equations (11) for the big cell. It is a direct check that in the variables

$$\tilde{p}_k = p_k - H_0^k, \quad k = 1, 2, 3, \dots \quad (26)$$

all equations (22) coincide with equations (11) for the big cell.

Thus the result for the stratum  $\Sigma_1$  and big cell are connected by a simple change of variables (26). Similar situation take place for other strata  $\Sigma_m$  with odd  $m$ .

## 4 Stratum $\Sigma_2$ and elliptic curves

For the stratum  $\Sigma_2$  with  $S = \{-2, 0, 2, 3, 4, \dots\}$  the positive order elements of the canonical basis are given by

$$\begin{aligned} p_0 &= 1 + \sum_{k \geq 1} \frac{H_k^0}{z^k}, \\ p_j &= z^j + H_{-1}^j z + \sum_{k \geq 1} \frac{H_k^j}{z^k}, \quad k, j = 2, 3, 4, \dots \end{aligned} \quad (27)$$

First we note that  $(p_{-2})^2 \notin \langle p_i \rangle_{i=-2, 0, 2, 3, \dots}$  and the analogue of the Lemmas 2.1 and 3.1 is given by

**Lemma 4.1** *A set  $W_2$  of Laurent series (27) obey the equations*

$$p_j(z)p_k(z) = \sum_{l=0, 2, 3, \dots} C_{jk}^l p_l(z) \quad (28)$$

and the condition  $z^2 W_2 \subset W_2$  is satisfied if and only if

$$\begin{aligned} H_k^{2n} &= 0, \quad k = -1, 1, 2, 3, \dots, \quad n = 0, 1, 2, \dots, \\ H_k^{2n+1} &= 0, \quad n, k = 1, 2, 3, \dots \end{aligned} \quad (29)$$

and

$$\begin{aligned} H_{2(k+n)+1}^{2m+1} - H_{2k+1}^{2(m+n)+1} - \sum_{s=-1}^{n-2} H_{2s+1}^{2m+1} H_{2k+1}^{2(n-s)+1} &= 0, \\ H_{2(n+k)+1}^{2m+1} + H_{2(m+k)+1}^{2n+1} + \sum_{s=-1}^k H_{2s+1}^{2m+1} H_{2(l-s)-1}^{2n+1} &= 0. \end{aligned} \quad (30)$$

Constants  $C_{jk}^l$  are given by

$$\begin{aligned} C_{2n, 2m}^{2l} &= \delta_{m+n}^l, \\ C_{2n, 2m+1}^{2l+1} &= \delta_{m+n}^l + H_{2(n-l)-1}^{2m+1}, \\ C_{2n+1, 2m+1}^{2l} &= \delta_{m+n+1}^l + H_{2(m-l)+1}^{2n+1} + H_{2(n-l)+1}^{2m+1} + H_{-1}^{2n+1} H_{-1}^{2m+1} \delta_2^l \\ &\quad + (H_{-1}^{2n+1} H_1^{2m+1} + H_1^{2n+1} H_{-1}^{2m+1}) \delta_0^l. \end{aligned} \quad (31)$$

which imply

$$\begin{aligned}
p_{2n}p_{2m} &= p_{2(m+n)}, \\
p_{2n}p_{2m+1} &= p_{2(m+n)+1} + \sum_{k=-1}^{n-2} H_{2k+1}^{2m+1} p_{2(n-k)-1}, \\
p_{2n+1}p_{2m+1} &= p_{2(m+n+1)} + \sum_{k=-1}^m H_{2k+1}^{2n+1} p_{2(m-k)} + \sum_{k=-1}^n H_{2k+1}^{2m+1} p_{2(n-k)} \\
&\quad + H_{-1}^{2n+1} H_{-1}^{2m+1} p_2 + (H_{-1}^{2n+1} H_1^{2m+1} + H_1^{2n+1} H_{-1}^{2m+1})
\end{aligned} \tag{32}$$

and  $p_{2n} = z^{2n}$ ,  $n \geq 0$ .

As a consequence, one has

**Proposition 4.2** *The stratum  $\Sigma_2$  contains a maximal closed subset  $W_2$  whose elements are vector spaces with basis (27) and parameters  $H_k^j$  obeying the constraints (29), (30) and such that  $z^2 W_2 \subset W_2$ .*

The relations (28) readily imply that all  $p_i(z)$  are generated by two elements  $z^2$  and  $p_3$ .

Using (32) one can show that the set of independent relations (28) is given by

$$p_3^2 = \lambda^3 + 2H_{-1}^3 \lambda^2 + (H_{-1}^3{}^2 + 2H_1^3) \lambda + 2H_{-1}^3 H_1^3 + 2H_3^3 \tag{33}$$

and

$$p_{2n+1} = \left( \lambda^{n-1} - \sum_{i=0}^{n-2} H_{-1}^{2(n-i)-1} \lambda^i \right) p_3 \tag{34}$$

This relation is obtained using iteratively the formula

$$p_{2n+1} = \lambda p_{2n-1} + H_{-1}^{2n-1} p_3. \tag{35}$$

**Proposition 4.3** *Subset  $W_2$  is an infinite family of infinite-dimensional commutative associative algebra with the basis  $1, p_2, p_3, p_4, \dots$  isomorphic to  $\mathbb{C}[\lambda, p_3]/C_6$*

where

$$C_6 = p_3^2 - \lambda^3 - 2H_{-1}^3 \lambda^2 - (H_{-1}^3{}^2 + 2H_1^3) \lambda - (2H_{-1}^3 H_1^3 + 2H_3^3). \tag{36}$$

**Proof** Associativity follows from the fact that the conditions (29) and (30) are equivalent to the condition

$$\sum_{s=0,2,3,\dots} C_{jk}^s C_{sl}^r = \sum_{s=0,2,3,\dots} C_{lk}^s C_{sj}^r \quad j, k, l, r = 0, 2, 3, \dots \tag{37}$$

for the constants  $C_{jk}^l$  given by (32).  $\square$

Treating now  $\lambda, p_3, p_5$  and  $H_k^j$  as affine coordinates one has the following geometrical interpretation of the subset  $W_2$ .

**Proposition 4.4** *The subset  $W_2$  is an infinite dimensional algebraic variety  $\Gamma_2$  in the affine space with coordinates  $p_j$ , ( $j = 2, 3, 4, \dots$ ),  $H_k^j$ , ( $j = 3, 5, 7, \dots$ ,  $k = -1, 1, 3, 5, \dots$ ) defined by the intersection of quadrics*

$$f_{jk} = p_j p_k - p_{j+k} - \sum_{l=0,2,3,\dots}^{j+k-1} C_{jk}^l p_l(z) = 0 \tag{38}$$

and quadrics (30). An ideal  $I^{(2)}$  of this variety is

$$I^{(2)} = \langle C_6, l_5^{(2)}, l_7^{(2)}, l_9^{(2)}, \dots \rangle \tag{39}$$

where  $l_{2n+1}^{(2)} = p_{2n+1} - \left( \lambda^{n-1} - \sum_{i=0}^{n-2} H_{-1}^{2(n-i)-1} \lambda^i \right) p_3$ .

Since  $W_2 \sim \mathbb{C}[\lambda, p_3]/C_6$  one can view  $\Gamma_2$  as the infinite family of coordinate rings of the elliptic curve  $C_6 = 0$  parameterized by the variables  $H_k^j$  obeying the conditions (29) and (30). Analyzing these conditions one concludes that there is an infinite set of independent variables among all  $H_k^j$ , for example  $H_{-1}^3, H_1^3, H_3^3, H_5^3, \dots$ .

It is a direct check that the curve  $C_6 = 0$  has genus one. So the stratum  $\Sigma_2$  contains an infinite family of elliptic curves parameterized by  $H_{-1}^3, H_1^3, H_3^3$ .

We emphasize that each of these elliptic curves belong to a point of the subset  $W_2$ . So, following [4], such points of  $\Sigma_2$  will be called *elliptic points* and the whole subset  $W_2$  an *elliptic subset*.

The ideal  $I^{(2)}$  contains singular hyperelliptic curves of all orders and of genus 1 given by

$$p_{2n+1}^2 = \left( \lambda^{n-1} - \sum_{i=0}^{n-2} H_{-1}^{2(n-i)-1} \lambda^i \right)^2 \left( \lambda^3 + 2H_{-1}^3 \lambda^2 + \left( H_{-1}^3{}^2 + 2H_1^3 \right) \lambda + 2H_{-1}^3 H_1^3 + 2H_3^3 \right) \quad (40)$$

## 5 Stratum $\Sigma_3$ : (3,4) curves of zero genus

Next case corresponds to  $S = \{-3, -1, 1, 3, 4, 5, \dots\}$ . Due to the absence of elements of orders zero and two positive elements of the canonical basis are given by

$$\begin{aligned} p_1 &= z + H_0^j + \sum_{k \geq 1} \frac{H_k^1}{z^k}, \\ p_j &= z^j + H_{-2}^j z^2 + H_0^j + \sum_{k \geq 1} \frac{H_k^j}{z^k}, \quad j = 3, 4, 5, \dots \end{aligned} \quad (41)$$

Since  $p_1^2$  has order two a closed subspace can be generated only by the elements  $p_3, p_4, p_5, \dots$ .

**Lemma 5.1** *A set  $W_3$  of Laurent series  $p_j(z)$ ,  $j = 3, 4, 5, \dots$  obey the equations*

$$p_j(z)p_k(z) = \sum_{l=3,4,5,\dots} C_{jk}^l p_l(z), \quad j, k = 3, 4, 5, \dots \quad (42)$$

and the condition  $z^2 W_3 \subset W_3$  if and only if

$$p_j = z^j + H_{-2}^j z^2 + H_0^j, \quad j \geq 5, \quad (43)$$

$$\begin{aligned} H_{-2}^j + H_{-2}^{j-2} H_{-2}^4 - H_0^{j-2} &= 0, \\ H_0^j + H_{-2}^{j-2} H_0^4 &= 0 \end{aligned} \quad (44)$$

and

$$\begin{aligned} H_0^4 + 2 H_0^3 H_{-2}^3 - H_{-2}^3{}^2 H_{-2}^4 - H_{-2}^4{}^2 &= 0, \\ H_0^3{}^2 - H_{-2}^3{}^2 H_0^4 - H_{-2}^4 H_0^4 &= 0. \end{aligned} \quad (45)$$

**Proof** Let us begin with the condition  $z^{2n} W_3 \subset W_3$ . One has

$$z^{2n} p_m(z) = z^{2n+m} + \dots + H_{2n-1}^m z + \dots \quad (46)$$

In  $W_3$  there is no element which contains the term of order one. Hence, with necessity  $H_{2n-1}^m = 0$  for all  $n = 1, 2, 3, \dots$  and  $m = 3, 4, 5, \dots$ , i.e.

$$p_j(z) = z^j + H_{-2}^j z^2 + H_0^j + \sum_{n \geq 1} \frac{H_{2n}^j}{z^{2n}}, \quad j = 3, 4, 5, \dots \quad (47)$$

Then considering the product  $p_{2k+1} p_j$  one has

$$p_{2k+1}(z) p_j(z) = z^{2k+j+1} + \dots + H_{2k}^j z + \dots \quad (48)$$

The terms of the order  $z^i$ ,  $i \geq 3$  can be represented as a superposition of  $p_3, p_4, \dots, p_{2k+j+1}$  giving the constants  $C_{jk}^l$  while the coefficient in front of  $z$  should vanish. Hence  $H_{2k}^j = 0$  for all  $k = 1, 2, 3, \dots$ . So

$$p_j = z^j + H_{-2}^j z^2 + H_0^j \quad j \geq 3. \quad (49)$$

The coefficients  $H_{-2}^j$  and  $H_0^j$  are not all independent. Indeed, the relations

$$\begin{aligned} z^2 p_3 &= p_5 + H_{-2}^3 p_4, \\ z^2 p_4 &= p_6 + H_{-2}^4 p_4, \\ z^4 p_3 &= p_7 + H_{-2}^3 p_6 + H_0^3 p_4, \\ &\dots \end{aligned} \quad (50)$$

imply

$$\begin{aligned} H_{-2}^5 - H_0^3 + H_{-2}^3 H_{-2}^4 &= 0, \\ H_0^5 + H_{-2}^3 H_0^4 &= 0, \\ H_{-2}^6 - H_0^4 + H_{-2}^4 H_{-2}^4 &= 0, \\ H_0^6 + H_{-2}^4 H_0^4 &= 0, \\ H_{-2}^7 + H_{-2}^3 H_{-2}^6 + H_0^3 H_{-2}^4 &= 0, \\ H_0^7 + H_{-2}^3 H_0^6 + H_0^3 H_0^4 &= 0, \\ &\dots \end{aligned} \quad (51)$$

and so on. The relations (51) are the lowest members of the relations (44). Using these relations, one can express all  $H_{-2}^j$ ,  $H_0^j$  with  $j = 5, 6, 7, \dots$  in terms of  $H_{-2}^3$ ,  $H_0^3$  and  $H_{-2}^4$ ,  $H_0^4$ .

Furthermore, the vanishing of the coefficients in front of  $z^2$  and  $z^0$  in the relation

$$p_3^2 - \left( p_6 + 2H_{-2}^3 p_5 + H_{-2}^3 p_4 + 2H_0^3 p_3 \right) = 0 \quad (52)$$

is equivalent to the conditions

$$\begin{aligned} H_{-2}^6 - 2H_{-2}^3 H_{-2}^5 - H_{-2}^3 H_{-2}^4 &= 0, \\ H_0^6 - H_0^3 H_{-2}^5 - H_{-2}^3 H_0^4 &= 0. \end{aligned} \quad (53)$$

Finally taking into account (51), one gets the constraints (45). So there are only two independent parameters among all coefficients  $H_{-2}^j$  and  $H_0^j$ . The simplest choice is to take  $H_{-2}^3$  and  $H_{-2}^4$  as independent variables. At last, the direct calculation gives

$$C_{jk}^l = \delta_{j+k}^l + H_{-2}^k \delta_{j+2}^l + H_0^k \delta_j^l + H_{-2}^j \delta_{k+2}^l + H_0^j \delta_k^l + H_{-2}^j H_{-2}^k \delta_4^l. \quad \square \quad (54)$$

An immediate consequence of the Lemma 5.1 is given by

**Proposition 5.2** *The stratum  $\Sigma_3$  contains the subset  $W_3$  closed with respect to pointwise multiplication  $W_3 \cdot W_3 \subset W_3$ . Elements of  $W_3$  are vector spaces with basis  $\langle p_i \rangle_i$  of the form (43) with  $H_{-2}^j, H_0^j$  obeying the constraints (44) and (45). The subset  $W_3$  is an infinite family of infinite-dimensional associative and commutative algebra  $A_3$  with the basis (43) and structure constants (54).*

A geometrical interpretation of  $W_3$  is provided by

**Proposition 5.3** *The subset  $W_3$  can be viewed as the two parametric family of algebraic varieties defined by the relations*

$$p_j p_k - \sum_l C_{jk}^l p_l = p_j p_k - \left( p_{j+k} + H_{-2}^k p_{j+2} + H_0^k p_j + H_{-2}^j p_{k+2} + H_0^j p_k + H_{-2}^j H_{-2}^k p_4 \right) = 0. \quad (55)$$



The ideal of this family contains the plane (3,4) curve (in the terminology of [7]) defined by the equation

$$\begin{aligned}
& p_4^3 - p_3^4 + 4 H_{-2}^3 p_3 p_4^2 - \left( 3 H_{-2}^4 - 2 H_{-2}^{3^2} \right) p_3^2 p_4 - \left( -4 H_0^3 + 2 H_{-2}^3 H_{-2}^4 \right) p_3^3 \\
& - \left( 3 H_0^4 + 4 H_0^3 H_{-2}^3 + H_{-2}^{3^4} + H_{-2}^{3^2} H_{-2}^4 \right) p_4^2 - \left( 4 H_{-2}^{3^2} H_0^3 + 8 H_{-2}^3 H_0^4 - 2 H_{-2}^3 H_{-2}^4 \right. \\
& \left. - 6 H_0^3 H_{-2}^4 - 2 H_{-2}^{3^3} H_{-2}^4 \right) p_3 p_4 - \left( -3 H_0^4 H_{-2}^4 + 6 H_0^{3^2} - 6 H_0^3 H_{-2}^3 H_{-2}^4 + 2 H_{-2}^{3^2} H_0^4 \right. \\
& \left. + H_{-2}^{3^2} H_{-2}^4 + H_{-2}^{4^3} \right) p_3^2 - \left( -3 H_0^{4^2} - 2 H_0^{3^2} H_{-2}^{3^2} - 2 H_{-2}^{3^2} H_0^4 H_{-2}^4 - 2 H_{-2}^{3^4} H_0^4 \right. \\
& \left. + 3 H_0^{3^2} H_{-2}^4 + 2 H_0^3 H_{-2}^3 H_{-2}^{3^2} - 8 H_0^3 H_{-2}^3 H_0^4 + 2 H_0^3 H_{-2}^{3^3} H_{-2}^4 \right) p_4 - \left( -4 H_0^{3^3} - 4 H_{-2}^3 H_0^{4^2} \right. \\
& \left. + 6 H_0^3 H_0^4 H_{-2}^4 - 4 H_0^3 H_0^4 H_{-2}^{3^2} + 6 H_0^{3^2} H_{-2}^3 H_{-2}^4 + 2 H_0^4 H_{-2}^3 H_{-2}^{3^2} + 2 H_0^4 H_{-2}^{3^3} H_{-2}^4 \right. \\
& \left. - 2 H_0^3 H_{-2}^{3^2} H_{-2}^{4^2} - 2 H_0^3 H_{-2}^{4^3} \right) p_3 = 0,
\end{aligned} \tag{56}$$

where  $H_{-2}^3, H_0^3, H_{-2}^4$  and  $H_0^4$  obey the constraints (45). The (3,4) curve (56) have zero genus.

**Proof** By direct calculation with the use of polynomial form (43) of  $p_j$ .  $\square$

Comparing the results of this and previous section, one observes an essential difference between the geometrical properties of the subspaces  $W_2$  and  $W_3$ . This is due to the quite different form of the Laurent series belonging to  $W_i$  which is the consequence of a different situation with elements of the first order in  $z$ . Namely, though in both cases  $W_i$  does not contain the element  $p_1(z)$ , The absence in  $W_3$  of the terms of order  $z$  in  $p_j(z)$  leads to a strong constraints leading to the polinomiality of  $p_j(z)$ .

We note also that due to the presence of the element  $p_0 = 1$  of zero order in  $W_2$  one has  $z^2 \in W_2$  while  $z^2 \notin W_3$ . As a consequence, for instance, one can choose  $p_3, p_4$  and  $z^2 p_3$  as the generators of the algebra  $A_3$  instead of  $p_3, p_4$  and  $p_5$ .

A way to avoid the polinomiality of all  $p_j(z) \in W_3$  would be to relax the condition  $z^2 W_3 \subset W_3$ . Since  $z^2$  is not an element of  $W_3$  it would be natural not to require that the product of  $z^2$  and an element of  $W_3$  belongs to  $W_3$ , but instead to require that  $z^2 W_3 \subset \Sigma_3$ . The presence of the element  $p_1(z)$  in  $\Sigma_3$ , allows us to avoid immediate constraints on  $p_j(z)$  followed from the relations of the type (46) and (48). for instance, instead of the conditions (50) one gets the following ones

$$\begin{aligned}
z^2 p_3 - p_5 - H_{-2}^3 p_4 &= H_1^3 p_1, \\
z^2 p_4 - p_6 - H_{-2}^4 p_4 &= H_1^4 p_1,
\end{aligned} \tag{57}$$

and so on. In virtue of the equations of this type one obtains an infinite set of relations for  $H_k^j$ . Computer analysis strongly indicates that these conditions again lead to the constraint  $H_k^j = 0$ ,  $k = 1, 2, 3, \dots$ ,  $j = 3, 4, 5, \dots$ , i.e. to the polinomiality of all  $p_j(z)$ .

## 6 Strata $\Sigma_{2n}$ . Hyperelliptic curves of genus $n$

Stratum  $\Sigma_{2n}$  with arbitrary  $n$  is characterized by  $S = \{-2n, -2n+2, -2n+4, \dots, 0, 2, 4, \dots, 2n, 2n+1, 2n+2, \dots\}$ . So it does not contain, in particular,  $n$  elements of the order  $1, 3, 5, \dots, 2n-1$  and the positive order elements of the canonical basis are given by

$$\begin{aligned}
p_0 &= 1 + \sum_{k \geq 1} \frac{H_k^0}{z^k}, \\
p_j &= z^j + \sum_{k=0}^{j-1} H_{-2k-1}^j z^{2k+1} + \sum_{k \geq 1} \frac{H_k^j}{z^k}, \quad j = 2, 4, 6, \dots, 2n-2, \\
p_j &= z^j + \sum_{k=0}^{n-1} H_{-2k-1}^j z^{2k+1} + \sum_{k \geq 1} \frac{H_k^j}{z^k} \quad j = 2n, 2n+1, 2n+2, 2n+3, \dots
\end{aligned} \tag{58}$$

As in the previous cases the  $p_j$  with negative  $j$  do not should be taked into account and one has

**Lemma 6.1** A set  $W_{2n}$  at fixed  $H_k^j$  of Laurent series (58) obey the condition  $z^2 W_{2n} \subset W_{2n}$  and equations

$$p_j(z)p_k(z) = \sum_l C_{jk}^l p_l(z), \quad j, k, l = 0, 2, 4, \dots, 2n, 2n+1, 2n+3, \dots \quad (59)$$

if and only if

$$\begin{aligned} H_k^{2m} &= 0, & m = 0, 1, 2, \dots, & k = -2n+2, -2n+4, \dots, -2, 0, 1, 2, 3, \dots, \\ H_{2k}^{2m+1} &= 0, & m = 0, 1, 2, \dots & k = -n, -n+1, -n+2, \dots \end{aligned} \quad (60)$$

and

$$\begin{aligned} H_{2(l+k)+1}^{2j+1} - H_{2l+1}^{2(j+k)+1} - \sum_{s=-n}^{k-1} H_{2s+1}^{2j+1} H_{2l+1}^{2(k-s)-1} &= 0, \\ H_{2(l+k)+1}^{2j+1} + H_{2(l+j)+1}^{2k+1} + \sum_{s=-n}^{-1} H_{2s+1}^{2j+1} H_{2(l-s)-1}^{2k+1} + \sum_{r=-n}^{-1} H_{2r+1}^{2k+1} H_{2(l-r)-1}^{2j+1} + \sum_{s=0}^{l-n} H_{2s+1}^{2j+1} H_{2(l-s)-1}^{2k+1} &= 0. \end{aligned} \quad (61)$$

Rewriting equation (59) separately for  $p_j$  with even and odd  $j$ , i.e.

$$\begin{aligned} p_{2j}p_{2k} &= p_{2(j+k)}, \\ p_{2j}p_{2k+1} &= p_{2(j+k)+1} + \sum_{s=-n}^{k-1} H_{2s+1}^{2j+1} p_{2(k-s)-1}, \\ p_{2j+1}p_{2k+1} &= p_{2(j+k)+1} + \sum_{s=-n}^j H_{2s+1}^{2j+1} p_{2(j-s)} + \sum_{s=-n}^k H_{2s+1}^{2k+1} p_{2(k-s)} \\ &\quad + \sum_{s=-n}^{-1} \sum_{r=-n}^{-1} H_{2s+1}^{2j+1} H_{2r+1}^{2k+1} p_{-2(s+r+1)} + \sum_{s=-n}^{-1} \sum_{r=0}^{-s-1} H_{2s+1}^{2j+1} H_{2r+1}^{2k+1} p_{-2(s+r+1)} \\ &\quad + \sum_{r=-n}^{-1} \sum_{s=0}^{-r-1} H_{2s+1}^{2j+1} H_{2r+1}^{2k+1} p_{-2(s+r+1)}, \end{aligned} \quad (62)$$

one concludes that

$$\begin{aligned} p_{2m} &= (z^2)^m, \\ p_{2m+1} &= \alpha(\lambda) p_{2n+1}, \quad m = n+1, n+2, \dots, \quad \lambda = z^2 \end{aligned} \quad (63)$$

for suitable  $\alpha(\lambda) \in \mathbb{C}[\lambda]$ . Moreover

$$p_{2n+1}^2 = \lambda^{2n+1} + \sum_{k=0}^{2n} u_k \lambda^k \quad (64)$$

where the coefficients  $u_k$  can be obtained from

$$p_{2n+1}^2 = \lambda^{2n+1} + 2 \sum_{s=0}^{2n} H_{2(n-s)+1}^{2n+1} \lambda^s + \sum_{k=-n}^{n+1} \sum_{s=0}^{n-k-1} H_{2k+1}^{2n+1} H_{-2(s+k)-1}^{2n+1} \lambda^s. \quad (65)$$

Thus, one has

**Proposition 6.2** The stratum  $\Sigma_{2n}$  for  $n = 2, 3, 4, \dots$  contains maximal subset  $W_{2n}$  closed with respect to pointwise multiplication. Elements of  $W_{2n}$  are vector spaces with basis given by  $\langle p_i \rangle_{i=0,2,4,\dots,2n,2n+1,2n+3,\dots}$  with parameters  $H_k^j$  obeying the constraints (60) and (61).

**Proposition 6.3** The subset  $W_{2n}$  is the infinite family of infinite-dimensional commutative associative algebra  $A_{2n}$  isomorphic to  $\mathbb{C}[\lambda, p_{2n+1}]/C_{2n+1}$  where  $\lambda = z^2$  and

$$C_{2n+1} = p_{2n+1}^2 - \lambda^{2n+1} - \sum_{k=0}^n u_k \lambda^k = 0 \quad (66)$$

and  $u_k$  are given by (65)

**Proof** Associativity of the algebra  $A_{2n}$  is the consequence of the equivalence of the constraints (60) and (61) and the associativity conditions

$$\sum_s C_{jk}^{is} C_{ls}^r = \sum_s C_{lk}^{is} C_{js}^r \quad (67)$$

for the constants  $C_{jk}^l$  defined in (59) i.e.

$$\begin{aligned} C_{2j,2k}^{2l} &= \delta_{j+k}^l, \\ C_{2j+1,2k}^{2l+1} &= \delta_{j+k}^l + H_{2(k-l)-1}^{2j+1}, \\ C_{2j,2k}^{2l} &= \delta_{j+k}^l + H_{2(j-l)+1}^{2j+1} + H_{2(k-l)+1}^{2k+1} + \sum_{s=-n-1}^{-1} \sum_{r=-n-1}^{-1} H_{2s+1}^{2j+1} H_{2r+1}^{2k+1} \delta_{-2(s+r+1)}^l \\ &\quad + \sum_{s=-n-1}^{-1} \sum_{r=0}^{-s-1} H_{2s+1}^{2j+1} H_{2r+1}^{2k+1} \delta_{-2(s+r+1)}^l + \sum_{r=-n-1}^{-1} \sum_{s=0}^{-r-1} H_{2s+1}^{2j+1} H_{2r+1}^{2k+1} \delta_{-(s+r+1)}^l. \end{aligned} \quad (68)$$

□

Interpreting  $\lambda, p_{2n+1}, p_{2n+3}, \dots$  as the local affine coordinates we first observe that the equation

$$C_{2n+1} = 0 \quad (69)$$

defines a hyperelliptic curve of genus  $n$ . It is parameterized by  $2n+1$  arbitrary quantities  $H_{-2n+1}^{2n+1}, H_{-2n+1}^{2n+3}, \dots, H_{+1}^{2n+1}, \dots, H_{2n+1}^{2n+1}$ . Their variation generates an infinite family of hyperelliptic curves.

Each hyperelliptic curve from this family (at fixed  $H_k^j$ ) is associated with a point of the subset  $W_{2n}$ . So one can refer to such points as *hyperelliptic points* in  $\Sigma_{2n}$  and the whole  $W_{2n}$  as *hyperelliptic subset* in  $\Sigma_{2n}$ .

One has

**Proposition 6.4** *The subset  $W_{2n}$  in  $\Sigma_{2n}$  is an infinite family of infinite-dimensional algebraic variety  $\Gamma_{2n}$  defined by the relations (59), (60), (61), (66). Its ideal is*

$$I_{2n+1} = \langle C_{2n+1}, l_{2n+3}^{(n)}, l_{2n+5}^{(n)}, \dots \rangle \quad (70)$$

where  $l_{2m+1}^{(n)} = p_{2m+1} - \alpha_m(\lambda)p_{2n+1}$ ,  $m = n+1, n+2, n+3, \dots$ .

In other words the variety  $\Gamma_{2n}$  is the intersection of the cubic  $C_{2n+1} = 0$  and infinite set of algebraic curves  $l_{2m+1}^{(n)}$ ,  $m = n+1, n+2, n+3, \dots$ . One can easily see that the ideal  $I_{2n}$  contains higher order hyperelliptic curves but all of them have genus  $n$ .

Thus stratum  $\Sigma_{2n}$  is characterized by the presence of the plane hyperelliptic curves  $C_{2n+1}$  of genus  $n$  in every point of the closed subset  $W_{2n}$ . This is due to the presence of  $n$  gaps (elements  $p_1(z), p_3(z), \dots, p_{2n-1}(z)$ ) in the basis of  $W_{2n}$ . The fact that for hyperelliptic curves (Riemann surfaces) of genus  $n$  one has  $n$  (Weierstrass) gaps in a generic point is well known in the theory of abelian functions (see e.g. [8]). Probably not that known observation is that these gaps and consequently the properties of corresponding algebraic curves are prescribed by the structure of the Birkhoff strata  $\Sigma_{2n}$  in  $\text{Gr}^{(2)}$ . In different context an appearance of hyperelliptic curves in Birkhoff strata of  $\text{Gr}^{(2)}$  has been observed in [9].

## 7 Strata $\Sigma_{2n+1}$

Stratum  $\Sigma_{2n+1}$ ,  $n = 2, 3, 4, \dots$  is characterized by  $S = \{-2n-1, -2n+1, \dots, -1, 1, 3, \dots, 2n+1, 2n+2, \dots\}$ . So, the positive order elements of the canonical basis in  $\Sigma_{2n+1}$  are of the form

$$\begin{aligned} p_j(z) &= z^j + H_{-j+1}^j z^{j-2} + H_{j+2}^j z^{j-2} + \dots + H_0^j + \sum_{k \geq 1} \frac{H_k^j}{z^k}, \quad j = 1, 3, \dots, 2n-1 \\ p_j(z) &= z^j + H_{-2n}^j z^{2n} + H_{-2n+1}^j z^{2n-1} + \dots + H_0^j + \sum_{k \geq 1} \frac{H_k^j}{z^k}, \quad j = 2n+1, 2n+2, \dots \end{aligned} \quad (71)$$

As in the previous cases the  $p_j$  with  $j \leq 1$  do not should be taken into account.

Closed subsets in  $\Sigma_{2n+1}$  have different structure for different  $n$ . In order to see this let us begin with  $\Sigma_5$ . In this case the elements (71) of the canonical basis are

$$\begin{aligned} p_1 &= z + H_0^1 + \sum_{k \geq 1} \frac{H_k^1}{z^k}, \\ p_3 &= z^3 + H_{-2}^3 z^2 + H_0^1 + \sum_{k \geq 1} \frac{H_k^1}{z^k}, \\ p_j &= z^j + H_{-4}^j z^4 + H_{-2}^j z^2 + H_0^j + \sum_{k \geq 1} \frac{H_k^j}{z^k} \quad j = 5, 6, 7, \dots \end{aligned} \quad (72)$$

It is easy to see that the maximal closed subset  $W_5$  in  $\Sigma_5$  is the subset whose points are vector spaces with basis  $(p_3, p_5, p_6, \dots)$ .

**Lemma 7.1** *A set  $W_5$  at fixed  $H_k^j$  of the Laurent series  $p_3, p_5, p_6, \dots$  obey the equations*

$$p_j(z)p_k(z) = \sum_{l=3,5,6,\dots} C_{jk} p_l(z) \quad (73)$$

and the condition  $z^2 W_5 \subset W_5$  if and only if  $H_k^j = 0$ ,  $j = 3, 5, 6, \dots$ ,  $k = 1, 2, 3, \dots$ , i.e. all  $p_j$  are polynomials and  $H_k^j$ ,  $k = -4, -2, 0$  obey the constraints

$$\begin{aligned} H_0^5 &= 0 & H_{-2}^5 &= H_0^3, & H_{-4}^5 &= H_{-2}^3, \\ H_0^6 &= -H_0^3{}^2, & H_{-2}^6 &= -2H_0^3 H_{-2}^3, & H_{-4}^6 &= -H_{-2}^3{}^2, \\ &\dots \end{aligned} \quad (74)$$

The proof of the polynomiality of  $p_j(z)$  is exactly the same as for  $W_3$  (Lemma 5.1). The constraints (74) follow from equations (73) and the condition  $z^5 W_5 \subset W_5$ . For instance one has  $z^2 p_3 = p_5$ ,  $z^2 p_5 = p_7 + H_{-4}^5 p_6$  etc. . The constants  $C_{jk}^l$  are given by

$$C_{jk}^l = \delta_{j+k}^l + \sum_{s=0}^2 H_{-2s}^j \delta_{2s+k}^l + \sum_{s=0}^2 H_{-2s}^k \delta_{2s+j}^l + \sum_{s,r=0}^2 H_{-2s}^j H_{-2r}^k \delta_{2(r+s)}^l, \quad j, k \geq 3 \quad (75)$$

where, for sake of compactess, we use  $H_{-4}^3 = 0$ . As a consequence of this lemma one has

**Proposition 7.2** *The stratum  $\Sigma_5$  contains a maximal subset  $W_5$  closed with respect to pointwise multiplication  $W_5 \cdot W_5 \subset W_5$ . Elements of  $W_5$  are vector spaces  $\langle p_i \rangle_{i=3,5,6,7,\dots}$  and  $H_{-4}^j$ ,  $H_{-2}^j$ ,  $H_0^j$  obeying the constraints (51).*

*Algebraically  $W_5$  is an infinite family of infinite-dimensional commutative associative algebra  $A_5$  of polynomials with the structure constants given by (75). Geometrically  $W_5$  is the infinite algebraic variety  $\Gamma_5$  defined by the equations (73) and (51).*

First equations of the set of equations (73) are

$$\begin{aligned} p_3^2 &= p_6 + 2H_{-2}^3 p_5 + 2H_0^3 p_3, \\ p_3 p_5 &= p_8 + 2H_{-2}^3 p_7 + H_{-2}^3{}^2 p_6 + 2H_0^3 p_5, \end{aligned} \quad (76)$$

and so on. So the algebra  $A_5$  is generated by  $p_3, p_5$  and  $p_7$ .

It is not also difficult to show that an ideal of the variety  $\Gamma_5$  contains the family of plane (3,5) curve

$$p_5^3 - p_3^5 + 2H_{-2}^3 p_3^3 p_5 - H_{-2}^3{}^2 p_3 p_5^2 + 2H_0^3 p_3^4 - 2H_0^3 H_{-2}^3 p_3^2 p_5 - H_0^3{}^2 p_3^3 = 0 \quad (77)$$

parameterized by two variables  $H_{-2}^3$  and  $H_0^3$ . Due to the polinomiality of  $p_3$  and  $p_5$  in  $z$ , the genus of of curve (77) is obviously equal to zero. The ideal of the varieties contains another rational plane curve

given by

$$\begin{aligned}
& p_6^5 - p_5^6 + 6 H_{-2}^3 p_5 p_6^4 + 14 H_{-2}^3 p_5^2 p_6^3 - \left( -6 H_0^3 - 16 H_{-2}^3 \right) p_5^3 p_6^2 - \left( -16 H_0^3 H_{-2}^3 - 9 H_{-2}^3 \right) p_5^4 p_6 \\
& - \left( -10 H_0^3 H_{-2}^3 - 2 H_{-2}^3 \right) p_5^5 + 2 H_0^3 p_6^4 + 8 H_0^3 H_{-2}^3 p_5 p_6^3 + 10 H_{-2}^3 H_0^3 p_5^2 p_6^2 \\
& - \left( -14 H_0^3 - 4 H_{-2}^3 H_0^3 \right) p_5^3 p_6 + 20 H_0^3 H_{-2}^3 p_5^4 + H_0^3 p_6^3 + 2 H_0^3 H_{-2}^3 p_5 p_6^2 + 8 H_0^3 p_5^3 = 0.
\end{aligned} \tag{78}$$

The stratum  $\Sigma_5$  exhibits the main features of higher strata  $\Sigma_{4m-1}$ , ( $m = 1, 2, 3, \dots$ ). The maximal closed subsets  $W_{4m-1}$  have the basis  $(p_{2m+1}, p_{2m+3}, \dots, p_{4m-1}, p_{4m}, p_{4m+1}, \dots)$  while the stratum  $\Sigma_{4m+1}$ , ( $m = 1, 2, 3, \dots$ ) have the basis  $(p_{2m+1}, p_{2m+3}, \dots, p_{4m-1}, p_{4m+1}, p_{4m+2}, \dots)$  with the respective  $p_j$ . Then one can demonstrate an analog of the Lemma 7.1 for  $\Sigma_{4m\pm 1}$  which in particular says that all  $p_j(z)$  are polynomials in  $z$  obeying the equations

$$p_j(z)p_k(z) = \sum_l C_{jk}^l p_l(z), \quad j, k, l = 2m+1, 2m+3, \dots \tag{79}$$

together with certain constraints on  $H_{-k}^j$ .

Consequently one has closed subsets  $W_{4m\pm 1}$  in  $\Sigma_{4m\pm 1}$  which algebraically are commutative and associative algebras and geometrically they represent families of algebraic varieties  $\Gamma_{4m\pm 1}$  defined by the equation (79). Ideals of the varieties  $\Gamma_{4m\pm 1}$  contain plane  $(2m+1, 2m+3)$  curve

$$p_{2m+1}^{2m+3} - p_{2m+3}^{2m+1} + \dots = 0, \quad m = 1, 2, 3, \dots \tag{80}$$

of zero genus.

Properties of these rational curves will be discussed elsewhere.

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